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Pancyclic Graphs I*

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A graph G with vertex set $V(G)$ and edge set $E(G)$ is *pancyclic* if it contains cycles of all lengths l, $3 \leqslant l \leqslant |V(G)|$.

Theorem. *Let G be Hamiltonian and suppose that $|E(G)| \geqslant n^2/4$, where $n = |V(G)|$. Then G is either pancyclic or else is the complete bipartite graph $K_{n/2,n/2}$.*

As a corollary to this theorem it is shown that the Ore conditions for a graph to be Hamiltonian actually imply that the graph is either pancyclic or else is $K_{n/2,n/2}$.

1. Introduction

Let G be a finite, undirected graph of order greater than 2, with no loops or multiple edges. We denote by $V(G)$, $E(G)$, respectively, the sets of vertices and edges of G, and by $d(v)$ the degree of vertex v in G. G is called *Hamiltonian* if it contains a cycle of length $|V(G)|$.

Various sufficient conditions for a graph to be Hamiltonian have been given in terms of the vertex degrees of the graph. One such is the following, due to Ore [4]: if

$$(u, v) \notin E(G) \Rightarrow d(u) + d(v) \geqslant |V(G)| \tag{1}$$

then G is Hamiltonian. We show in the next section that condition (1) implies considerably more about the cycle structure of the graph G.

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2. Pancyclic Graphs

We call G *pancyclic* if there are cycles of each length l, $3 \leqslant l \leqslant |V(G)|$, in G. Pancyclic digraphs—that is, digraphs with directed cycles (circuits) of all lengths—have been studied for the particular case of tournaments. Harary and Moser [2] proved that strong tournaments are pancyclic and Moon [3] showed further that every vertex of a strong tournament is in a circuit of each length. Alspach [1] proved that in a regular tournament every edge features in a circuit of each length.

From the definition it follows that a pancyclic graph is Hamiltonian. We here give a condition under which the converse holds.

Theorem. *Let G be Hamiltonian and suppose that $|E(G)| \geqslant n^2/4$, where $n = |V(G)|$. Then G is either pancyclic or else is the complete bipartite graph $K_{n/2,n/2}$.*

Proof. Let $C \equiv (v_1, v_2, ..., v_n)$ be a cycle of length n in G. (We represent C by its vertices in sequential order.) Each edge of G can be regarded as a chord of C. We define the *length* of a chord to be the distance between its endpoints, taken round C.

Assume that G is not pancyclic. Then G has no cycle of some length l, $3 \leqslant l < n$. For adjacent vertices v_j, v_{j+1} on C, consider the following pairing of chords of C (*Note*: here, as in the rest of this section, all suffices should be taken modulo n):

For $j + l - 1 \leqslant k \leqslant j - 1$ pair (v_j, v_k) with (v_{j+1}, v_{k-l+3}).

For $j + 2 \leqslant k \leqslant j + l - 2$ pair (v_j, v_k) with (v_{j+1}, v_{k-l+1}).

Then it is clear that at most one chord of each pair can be an edge of G, since both chords of a pair, together with part of C, make up a cycle of length l (see Figure 1).

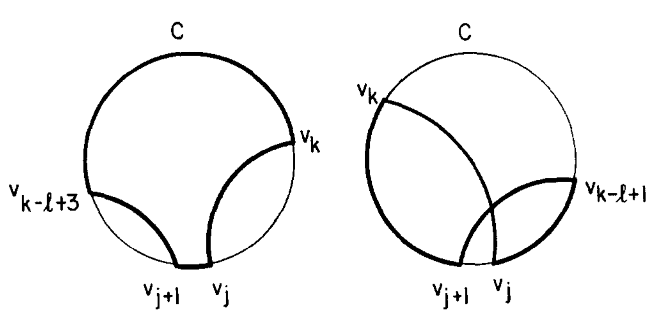

Figure 1

It follows that, for every j,

$$d(v_j) + d(v_{j+1}) \leqslant n \tag{2}$$

with equality if and only if exactly one of each pair of chords is an edge of G.

We first show that n must be even. For suppose n is odd. Then, by (2), some vertex, which we can take to be v_n, has degree at most $(n-1)/2$. Hence, again by (2),

$$2 \mid E(G)| = \sum_{j=1}^{n} d(v_j) \leqslant \frac{n(n-1)}{2} + d(v_n) \leqslant \frac{(n-1)(n+1)}{2} < \frac{n^2}{2}.$$

Therefore G has fewer than $n^2/4$ edges, contrary to hypothesis. Since n is even it follows from (2) that

$$| E(G)| \leqslant n^2/4.$$

But by hypothesis

$$| E(G)| \geqslant n^2/4.$$

Hence $| E(G)| = n^2/4$, and equality is attained in (2) for each j. Therefore

$$(v_j, v_k) \in E(G) \Leftrightarrow (v_{j+1}, v_{k-l+3}) \notin E(G), \qquad j+l-1 \leqslant k \leqslant j-1; \tag{3}$$

$$(v_j, v_k) \in E(G) \Leftrightarrow (v_{j+1}, v_{k-l+1}) \notin E(G), \qquad j+2 \leqslant k \leqslant j+l-2. \tag{4}$$

Suppose that G is not the complete bipartite graph $K_{n/2,n/2}$. Then some edge of G is a chord of even length. We shall show that there is a chord of length 2 in G. For assume that $k \geqslant 4$ is the shortest even chord length in G, and that $(v_j, v_{j+k}) \in E(G)$. There are three cases, each of which is illustrated in Figure 2.

(i) $4 \leqslant k \leqslant n-l$.

$(v_{j+1}, v_{j+k+l-3}) \notin E(G)$ since this chord, together with (v_j, v_{j+k}) and part of C, forms a cycle of length l. Hence, by (3), $(v_{j+2}, v_{j+k}) \in E(G)$ and this chord has length $k-2$, contradicting the minimality of k.

(ii) $n-l+2 \leqslant k \leqslant 2n-2l$.

By (4) $(v_{j-1}, v_{j+k+l-1}) \notin E(G)$; by (3) $(v_{j-2}, v_{j+k+2l-4}) \in E(G)$. But this is a chord of length $2n-k-2l+2 \leqslant k-2$, and we again have a contradiction.

(iii) $2n-2l+2 \leqslant k \leqslant n-2$.

By (4) $(v_{j-1}, v_{j+k+l-1}) \notin E(G)$ and, again by (4), $(v_{j-2}, v_{j+k+2l-2}) \in E(G)$.

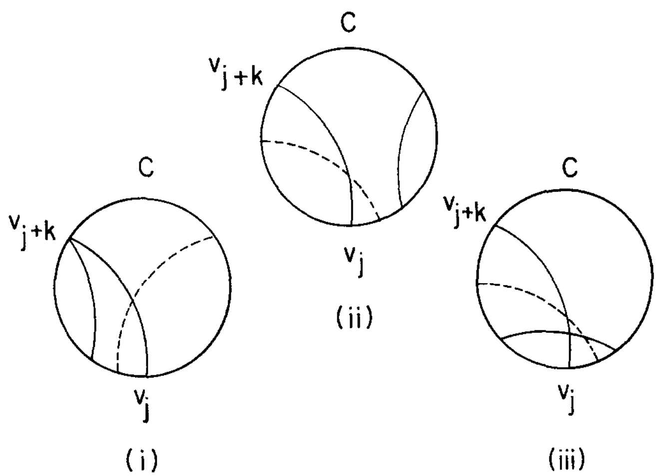

FIGURE 2

This chord has length $k + 2l - 2n \leqslant k - 2$, once more contradicting the minimality of k.

It follows that there is a chord $(v_j, v_{j+2}) \in E(G)$. But then $(v_j, v_{j+l}) \notin E(G)$ and so, by (3), $(v_{j+1}, v_{j+3}) \in E(G)$. Continuing round C in this way we conclude that all chords of length 2 are in G. But then it is clear that G is pancyclic. This is contrary to assumption and so G must indeed be $K_{n/2,n/2}$.

COROLLARY. *Let G be a graph satisfying condition* (1). *Then G is either pancyclic or else is the complete bipartite graph $K_{n/2,n/2}$.*

Proof. In view of Ore's theorem it is only necessary to show that condition (1) implies $|E(G)| \geqslant n^2/4$. Let k be the minimum vertex degree in G. We assume that $k < n/2$, for otherwise there is nothing to prove. If there are m vertices of degree k then, by (1), these vertices must all be joined to one another. Hence $m \leqslant k + 1$; $m \neq k + 1$ since G is Hamiltonian and therefore connected. There are at least $n - k - 1$ vertices of degree at least $n - k$ in G, namely those vertices not joined to one specific vertex of degree k. Hence

$$
\begin{aligned}
|E(G)| &\geqslant \tfrac{1}{2}\{(n-k-1)(n-k) + k^2 + k + 1\} \\
&= \tfrac{1}{2}\{n^2 - n(2k+1) + 2k^2 + 2k + 1\} \\
&\geqslant \frac{n^2+1}{4}.
\end{aligned}
$$

3. Conclusion

Apart from Ore's theorem, several other conditions for a graph or digraph to be Hamiltonian are known. It seems likely that results similar to the Corollary of §2 will also hold for some of these conditions. With this in mind we offer the following two conjectures.

CONJECTURE 1. *Let G be a Hamiltonian digraph of order n with no loops or multiple edges, and with $|E(G)| \geqslant n^2/2$. Then either G is pancyclic or else $G \simeq \overrightarrow{K}_{n/2,n/2}$, where $\overrightarrow{K}_{n/2,n/2}$ is the symmetric digraph associated with $K_{n/2,n/2}$.*

CONJECTURE 2. *Let G be a 4-connected planar graph. Then G is pancyclic.*†

Finally we mention one extremal problem concerning pancyclic graphs. What is the minimum number of edges in a pancyclic graph of order n? If this number is denoted by $p(n)$ we can prove that, for $n \geqslant 3$,

$$n - 1 + \log_2(n - 1) \leqslant p(n) \leqslant n + \log_2(n) + H(n) + 0(1),$$

where $H(n)$ is the smallest integer such that $(\log_2)^{H(n)}(n) < 2$.

Acknowledgments

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† *Added in proof.* J. Malkevitch has recently constructed examples of 4-connected planar graphs without 4-cycles, ('On the lengths of cycles in planar graphs', Proceedings of the Conference on Graph Theory and Combinatorics, St. John's University, Jamaica, New York, 1970).

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